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Generalized Barut-Girardello Coherent States for Mixed States with Arbitrary Distribution

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Abstract In the paper we examine some properties of the generalized coherent states of the Barut-Girardello kind. These states are defined as eigenstates of a generalized lowering operator and they are strongly dependent on the structure constants. Besides the pure coherent states we focused our attention on the mixed states one, which are characterized by different probability distributions. As some examples we consider the thermal canonical distribution and the Poisson distribution functions. We calculate for these cases the Husimi's Q and quasi-probability P-distribution functions.

Keywords Coherent states · Harmonic oscillator · Pseudoharmonic oscillator · Poisson distribution · Thermal distribution

1 Introduction

Besides the canonical coherent states (CSs) of the one-dimensional harmonic oscillator (HO-1D), many authors have been tried to build new CSs, i.e. CSs which correspond to an arbitrary physical Hamiltonian with known discrete eigenvalues (see, [1, 2] and references therein and also [3] and [4]).

Generically, the CSs that are not canonical were called *generalized coherent states*. It exists three formalisms to obtain the generalized CSs:

- as eigenvectors of a lowering operator A_- (this kind of CSs are called the Barut-Girardello CSs, BG-CSs);
- by applying a generalized (exponential) displacement operator on a ground (or fiducial) state (this kind of CSs are called the Klauder-Perelomov CSs, KP-CSs);
- by expressing the CSs through two real variables, say $J \in (0, +\infty)$ and $\gamma \in (-\infty, +\infty)$, the corresponding structure constants of CSs being dependent on the energy eigenvalues (this kind of CSs are called the Gazeau-Klauder CSs, GK-CSs).

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Only for the one dimensional harmonic oscillator these three types of coherent states are identical, for all other physical systems it exists the differences between CSs.

The aim of our paper will be only the generalized BG-CSs and some of their properties. We will examine the properties of these states by expressing the main elements (the weight function of the integration measure, the Husimi's Q-function, the P-quasi-distribution function and so on) through the Meijer's G-functions [5]. This way offer an elegant and quick possibility to build the generalized BG-CSs which correspond to different Hamiltonians with known discrete eigenvalues on the one hand and with arbitrary distribution functions of the mixed states on the other hand. As examples we have chosen two potentials with linear energy eigenvalues spectrum: the one dimensional harmonic oscillator and the pseudoharmonic oscillator, the last having the advantage to be an intermediate oscillator between the harmonic and the more anharmonic potentials (e.g. the Morse or Pöschl-Teller potentials). As regarded the distribution functions for mixed states we paid our attention to two cases: the thermal (canonical) and the Poisson distribution functions which are the most popular on quantum optics and also in the physics of quantum information.

2 General Considerations

Let us consider a set of Fock vectors $|n;\lambda\rangle$, with λ a parameter and a pair of raising A_+ and lowering A_- general operators which act on the vector basis as follows [1]:

$$A_-|n;\lambda\rangle = \sqrt{e_n(\lambda)}|n-1;\lambda\rangle$$
$$A_+|n;\lambda\rangle = \sqrt{e_{n+1}(\lambda)}|n+1;\lambda\rangle,$$

so that

$$A_+A_-|n;\lambda\rangle = \sqrt{e_n(\lambda)}|n;\lambda\rangle. \tag{2}$$

The most general coherent states (CSs) of the Barut-Girardello kind (i.e. BG-CSs) can be defined through the equation [2]:

$$A_-|z;\lambda\rangle = z|z;\lambda\rangle, \tag{3}$$

where $z = |z|\exp(i\varphi)$ is a complex variable.

Because the Fock states provide a basis set for amplitude space, the CSs can be expressed as a sum over the Fock states. Consequently, the BG-CSs $|z;\lambda\rangle$ can also be expanded in the Fock basis as:

$$\begin{aligned}|z;\lambda\rangle &= \sum_{n=0}^{\infty}\langle n;\lambda \mid z;\lambda\rangle|n;\lambda\rangle \\ &\equiv \sum_{n=0}^{\infty} c_n\big(|z|^2;\lambda\big)|n;\lambda\rangle.\end{aligned} \tag{4}$$

In order to calculate the functions $c_n(|z|^2;\lambda)$ we will act with the operator A_- on both sides of the previous equation using the definition of the BG-CSs and the eigenequation of the lowering operator:

$$z\sum_{n=0}^{\infty} c_n\big(|z|^2;\lambda\big)|n;\lambda\rangle = \sum_{n=0}^{\infty} c_n\big(|z|^2;\lambda\big)\sqrt{e_n(\lambda)}|n-1;\lambda\rangle. \tag{5}$$

Then, we will multiply the above equation with the bra vector $\langle m;\lambda|$ considering the normalization condition for the Fock vectors:

$$\langle m;\lambda \mid n;\lambda\rangle = \delta_{mn}. \tag{6}$$

We obtain, successively:

$$zc_m\big(|z|^2;\lambda\big) = c_{m+1}\big(|z|^2;\lambda\big)\sqrt{e_{m+1}(\lambda)}. \tag{7}$$

This recurrence relation leads to the following equation for the functions $c_n(|z|^2;\lambda)$:

$$\begin{aligned} c_n\big(|z|^2;\lambda\big) &= c_0\big(|z|^2;\lambda\big)\frac{z^n}{\sqrt{e_1(\lambda)e_2(\lambda)\cdots e_n(\lambda)}} \\ &\equiv c_0\big(|z|^2;\lambda\big)\frac{z^n}{\sqrt{\rho_n(\lambda)}}. \end{aligned} \tag{8}$$

The new introduced functions, with respect to the parameter λ, i.e.

$$\rho_n(\lambda) \equiv e_1(\lambda)e_2(\lambda)\cdots e_n(\lambda) = \prod_{j=1}^{n} e_j(\lambda), \tag{9}$$

are called the structure constants of the coherent state, because they play a decisive role in the expression of CSs, i.e. they entirely determine the expression of CSs.

Finally, the expansion of the CSs in the Fock basis becomes:

$$|z;\lambda\rangle = c_0\big(|z|^2;\lambda\big)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\rho_n(\lambda)}}|n;\lambda\rangle. \tag{10}$$

The normalization function $c_0(|z|^2;\lambda)$ is determined as a consequence of the normalization condition of the CSs:

$$\langle z;\lambda \mid z;\lambda\rangle = 1 \tag{11}$$

and it can be read as follows:

$$\begin{aligned} c_0\big(|z|^2;\lambda\big) &= \left[\sum_{n=0}^{\infty}\frac{(|z|^2)^n}{\rho_n(\lambda)}\right]^{-\frac{1}{2}} \\ &\equiv \left[S^{(0)}\big(|z|^2;\lambda\big)\right]^{-\frac{1}{2}}, \end{aligned} \tag{12}$$

where we have used a special notation for the sum from the square brackets, which will be useful for further determinations.

In some cases of CSs, e.g. the BG-CSs of the one dimensional harmonic oscillator (HO-1D) or pseudoharmonic oscillator (PHO), we can express the normalization function $c_0(|z|^2;\lambda)$ through elementary analytical functions, but, she generally is given through the hypergeometric functions ${}_pF_q(\{a_p\}; \{b_q\}; |z|^2)$ where $\{a_p\} \equiv a_1, a_2, \ldots, a_p$ and $\{b_q\} \equiv b_1, b_2, \ldots, b_q$ are real coefficients [3].

The BG-CSs $|z; \lambda\rangle$ are normalizable but non orthogonal:

$$\langle z; \lambda \mid z'; \lambda\rangle = \frac{c_0(|z|^2; \lambda) c_0(|z'|^2; \lambda)}{c_0(z' z^*; \lambda)}. \tag{13}$$

According to Klauder's minimal prescriptions, the coherent states must also allow the identity operator resolution [4]:

$$\int d\mu(z; \lambda) |z; \lambda\rangle\langle z; \lambda| = 1. \tag{14}$$

The integration measure must necessarily have the following structure:

$$d\mu(z; \lambda) \equiv \frac{d^2 z}{\pi} h\big(|z|^2\big) = \frac{d\varphi}{2\pi} d\big(|z|^2\big) h\big(|z|^2\big). \tag{15}$$

By inserting this expression in the equation of the operator resolution, we have:

$$\int_0^{2\pi} \frac{d\varphi}{2\pi} \int_0^R d\left(|z|^2\right) h\left(|z|^2; \lambda\right) \left[c_0\left(|z|^2; \lambda\right)\right]^2 \times \sum_{m,n=0}^{\infty} \frac{(z^*)^m z^n}{\sqrt{\rho_m(\lambda)\rho_n(\lambda)}} |n; \lambda\rangle\langle m; \lambda| = 1, \tag{16}$$

where R is the convergence radius.

Due to the fact that the angular integral is

$$\int_0^{2\pi} \frac{d\varphi}{2\pi} \left(z^*\right)^m z^n = \left(|z|^2\right)^n \delta_{mn}, \tag{17}$$

we perform the function change:

$$g\left(|z|^2; \lambda\right) \equiv h\left(|z|^2; \lambda\right) \left[c_0\left(|z|^2 \lambda\right)\right]^2 \tag{18}$$

and the above expression leads to:

$$\int_0^R d\left(|z|^2\right) \left(|z|^2\right)^n g\left(|z|^2; \lambda\right) = \rho_n\left(\lambda\right). \tag{19}$$

This is a well-known Stieltjes (for the convergence radius $R \to \infty$) or Hausdorff (for $R < \infty$) moment problem [4] which leads to the CSs defined on the entire complex plane or on a complex circle with radius R.

The Barut-Girardello coherent states can be defined only for the quantum systems with infinite dimensional Hilbert spaces and, consequently, for these states the convergence radius is $R \to \infty$ and the states are defined on the entire complex plane.

This problem can be solved if we get from the real (entire) numbers n to the complex variable s, by identification $n \to s-1$ [4]. So, the last integral becomes:

$$\int_0^R d\left(|z|^2\right)\left(|z|^2\right)^{s-1} g\left(|z|^2;\lambda\right) = \rho_{s-1}(\lambda)\,. \tag{20}$$

The general solution of this integral equation is the Meijer's G-function, i.e. [5]:

$$\begin{aligned} g\left(|z|^2;\lambda\right) &= G^{m_1 n_1}_{p_1 q_1}\left(|z|^2 \,\middle|\, \begin{matrix} a_1,\ldots,a_{n_1}; & a_{n_1+1},\ldots,a_{p_1} \\ b_1,\ldots,b_{m_1}; & b_{m_1+1},\ldots,b_{q_1}\end{matrix}\right) \\ &\equiv G^{m_1 n_1}_{p_1 q_1}\left(|z|^2 \,\middle|\, \begin{matrix} \{a_{n_1;p_1}\} \\ \{b_{m_1;q_1}\}\end{matrix}\right). \end{aligned} \tag{21}$$

Finally, the integration measures becomes:

$$d\mu(z;\lambda) = \frac{d^2 z}{\pi} S^{(0)}\left(|z|^2;\lambda\right) G^{m_1 n_1}_{p_1 q_1}\left(|z|^2 \,\middle|\, \begin{matrix} \{a_{n_1;p_1}\} \\ \{b_{m_1;q_1}\}\end{matrix}\right). \tag{22}$$

The integration measure must be expressed by positive weight functions. As a consequence, among all Meijer's functions which are the mathematical solutions of the integral equation, in order to construct the BG-CSs, we have to choose only these functions which ensure the positivity of the integration measure function.

Another condition for the weight function of the integration measure is its unicity. The unicity of the solution of the above integral equation can be tested by using the customary algebraic methods for the power series convergence. If we denote the generic term of a series with T_n, then depending on the value of the following sum, we can have two situations:

$$\begin{aligned} S &= \sum_{n=1}^{\infty} T_n \equiv \sum_{n=1}^{\infty} [\rho_n(\lambda)]^{-\frac{1}{2n}} \\ &= \begin{cases} \infty, & \text{it exists only one solution;} \\ <\infty, & \text{it exists many solutions.}\end{cases} \end{aligned} \tag{23}$$

In order to determine the convergence of this series we can use, e.g. the d'Alembert criterion of convergence:

$$\lim_{n\to\infty} \frac{\ln T_n}{\ln n} \begin{cases} <-1, & S \text{ is convergent;} \\ >-1, & S \text{ is divergent.}\end{cases} \tag{24}$$

As a partial conclusion we can say that if we know only the action of the lowering operator A_-, i.e. if we know the functions $e_j(\lambda)$ and, implicitly, the structure constants $\rho_n(\lambda)$, then we can certainly define the coherent states of the Barut-Girardello kind, with positive integration measure expressed through the Meijer's G-function.

A pure BG-CSs $| z;\lambda >$ can be characterized by the following density operator or projector onto the BG-CSs:

$$\begin{aligned} \rho_{|z;\lambda\rangle} &\equiv |z;\lambda\rangle\langle\lambda;z| \\ &= \left[c_0\left(|z|^2;\lambda\right)\right]^2 \sum_{m,n=0}^{\infty} \frac{(z^*)^m z^n}{\sqrt{\rho_m(\lambda)\rho_n(\lambda)}} |n;\lambda\rangle\langle m;\lambda|. \end{aligned} \tag{25}$$

This expression is useful in order to calculate the expectation value for an observable A (which characterizes the quantum system) in a pure BG-CS $|z;\lambda\rangle$:

$$\begin{aligned}\langle A\rangle_{|z;\lambda\rangle} &= \mathrm{Tr}\big(\rho_{|z;\lambda\rangle} A\big)\\ &= \int d\mu\left(z';\lambda\right)\langle z';\lambda|\rho_{|z;\lambda\langle} A|z';\lambda\rangle\\ &= \int d\mu\left(z';\lambda\right)\langle z';\lambda|z;\lambda\rangle\langle z;\lambda|A|z';\lambda\rangle\\ &= \langle z;\lambda|A\left(\int d\mu\left(z';\lambda\right)|z';\lambda\rangle\langle z';\lambda|\right)|z;\lambda\rangle\\ &= \langle z;\lambda|A|z;\lambda\rangle. \end{aligned} \tag{26}$$

Finally, we obtain:

$$\langle A\rangle_{|z;\lambda\rangle} = \left[c_0\left(|z|^2;\lambda\right)\right]^2 \sum_{m,n=0}^{\infty} \frac{(z^*)^m z^n}{\sqrt{\rho_m(\lambda)\rho_n(\lambda)}}\langle m;\lambda \mid A \mid n;\lambda\rangle. \tag{27}$$

Consequently, the density of probability of the transition between a Fock state $|n;\lambda\rangle$ and a BG-CS $|z;\lambda\rangle$ can be obtained if we calculate an expectation value of the projector $A_{|n;\lambda\rangle} = |n;\lambda\rangle\langle n;\lambda|$ in the Fock space:

$$\begin{aligned}P_{n;|z|^2} &\equiv \langle z;\lambda|A_{n;\lambda}|z;\lambda\rangle = |\langle n;\lambda \mid z;\lambda\rangle|^2\\ &= \left[c_0\left(|z|^2;\lambda\right)\right]^2 \frac{(|z|^2)^n}{\rho_n(\lambda)}. \end{aligned} \tag{28}$$

An interesting and, in the same time, useful operator is the particle number operator defined as

$$N|n;\lambda\rangle = n|n;\lambda\rangle. \tag{29}$$

The expectation value of their integer power s in the pure BG-CS $|z;\lambda\rangle$, i.e. $\langle N^s\rangle_{|z;\lambda\rangle}$ is:

$$\begin{aligned}\langle N^s\rangle_{|z;\lambda\rangle} &= \langle z;\lambda \mid N^s \mid z;\lambda\rangle\\ &= \left[c_0\left(|z|^2;\lambda\right)\right]^2 \sum_{n=0}^{\infty} \frac{(|z|^2)^n}{\rho_n(\lambda)} n^s. \end{aligned} \tag{30}$$

We observe that

$$\begin{aligned}S^{(s)}\left(|z|^2;\lambda\right) &\equiv \sum_{n=0}^{\infty} \frac{(|z|^2)^n}{\rho_n(\lambda)} n^s\\ &= \left(|z|^2 \frac{d}{d|z|^2}\right)^s \left[\sum_{n=0}^{\infty} \frac{(|z|^2)^n}{\rho_n(\lambda)}\right]\\ &= \left(|z|^2 \frac{d}{d|z|^2}\right)^s S^{(0)}\left(|z|^2;\lambda\right), \end{aligned} \tag{31}$$

so we have:

$$\langle N^s\rangle_{|z;\lambda\rangle} = \frac{1}{S^{(0)}(|z|^2;\lambda)}\left(|z|^2\frac{d}{d|z|^2}\right)^s S^{(0)}\left(|z|^2;\lambda\right). \tag{32}$$

Through the expectation values $\langle N^s\rangle_{|z;\lambda\rangle}$ we can express different quantities, e.g. the Mandel's Q parameter [6]. Various criteria have been used in the literature in order to quantify the non-classical character of a given quantum state [6]. One such a criterion was introduced by Mandel [7], defining the parameter

$$\begin{aligned} Q_{|z;\lambda\rangle} &= \frac{(\sigma_N^2)_{|z;\lambda\rangle}}{\langle N\rangle_{|z;\lambda\rangle}} - 1 \qquad (33)\\ &= \frac{\langle N^2\rangle_{|z;\lambda\rangle} - (\langle N\rangle_{|z;\lambda\rangle})^2}{\langle N\rangle_{|z;\lambda\rangle}} - 1. \end{aligned}$$

The Mandel's parameter determines the relative departure of the standard deviation $(\sigma_N^2)_{|z;\lambda\rangle}$ to the number operator expectation value $\langle N\rangle_{|z,\lambda\rangle}$. For the canonical coherent states of the one-dimensional harmonic oscillator (HO-1D) [6] this deviation is zero and, consequently, the distribution function of these coherent states is Poissonian.

As it is well known, the standard deviation $(\sigma_N^2)_{|z;\lambda\rangle}$ can be found from the variance according to

$$\left(\sigma_N^2\right)_{|z;\lambda\rangle} = \langle N^2\rangle_{|z;\lambda\rangle} - \left(\langle N\rangle_{|z;\lambda\rangle}\right)^2. \tag{34}$$

In other words, the Mandel's parameter quantifies the departure of the particle-number distribution function of a certain quantum state from the Poisson distribution function, i.e. a certain state-statistics to the Poisson statistics. If $Q_{|z;\lambda\rangle}\langle 0$ the field is called sub-Poissonian, if $Q_{|z;\lambda\rangle} = 0$ it is Poissonian, while if $Q_{|z;\lambda\rangle}\rangle 0$ the corresponding field is called supra-Poissonian.

After some simple calculations we obtain:

$$\begin{aligned} Q_{|z;\lambda\rangle} &= |z|^2\left[\frac{(\frac{d}{d|z|^2})^2 S^{(0)}(|z|^2;\lambda)}{(\frac{d}{d|z|^2})S^{(0)}(|z|^2;\lambda)} - \frac{(\frac{d}{d|z|^2})S^{(0)}(|z|^2;\lambda)}{S^{(0)}(|z|^2;\lambda)}\right]\\ &\equiv |z|^2\left[\frac{S''^{(0)}(|z|^2;\lambda)}{S'^{(0)}(|z|^2;\lambda)} - \frac{S'^{(0)}(|z|^2;\lambda)}{S^{(0)}(|z|^2;\lambda)}\right]. \qquad (35) \end{aligned}$$

3 Coherent States for Mixed States

Now, let us consider an arbitrary physical quantum system in a mixed quantum state which is characterized by the normalized density operator:

$$\rho(\lambda) = \sum_{n=0}^{\infty} w_n(\lambda)|n;\lambda\rangle\langle n;\lambda|. \tag{36}$$

For the moment we consider that the probability density of finding the system in a pure state $|n;\lambda\rangle$, i.e. $w_n(\lambda)$ is an arbitrary function of the energy quantum number n and, due to the normalization condition, we have:

$$\operatorname{Tr}\rho(\lambda) = \sum_{n=0}^{\infty} w_n(\lambda) = 1. \tag{37}$$

We are interested in the coherent states representation of the density operator:

$$\rho(\lambda) = \int d\mu(z;\lambda)|z;\lambda\rangle P\left(|z|^2;\lambda\right)\langle z;\lambda|. \tag{38}$$

In other words, we try to find the expression of the quasi-probability P-function $P(|z|^2;\lambda)$.

By inserting in this equation the expansions of CSs in the Fock basis vectors and also the integration measure, after the angular integration, we are lead to the expression:

$$\int_0^\infty d\left(|z|^2\right)\left(|z|^2\right)^n P\left(|z|^2;\lambda\right) G_{p_1q_1}^{m_1n_1}\left(|z|^2 \,\middle|\, \begin{matrix}\{a_{n_1;p_1}\}\\ \{b_{m_1;q_1}\}\end{matrix}\right) = w_n(\lambda)\rho_n(\lambda). \tag{39}$$

By passing to the complex variable and performing the function change:

$$R\left(|z|^2;\lambda\right) \equiv P\left(|z|^2;\lambda\right) G_{p_1q_1}^{m_1n_1}\left(|z|^2 \,\middle|\, \begin{matrix}\{a_{n_1;p_1}\}\\ \{b_{m_1;q_1}\}\end{matrix}\right), \tag{40}$$

we get to the expression:

$$\int_0^\infty d\left(|z|^2\right)\left(|z|^2\right)^n R\left(|z|^2;\lambda\right) = w_n(\lambda)\rho_n(\lambda). \tag{41}$$

This relative complicated structure of the function becomes much simpler if we go to the concrete quantum systems, as we can see below.

As an example, if the quantum system is spanned by an infinite dimensional Hilbert space, the energy spectra of such quantum systems are linear with respect to the energy quantum number n and, consequently, the energy spectrum has the following structure:

$$E_n = an + b, \tag{42}$$

$$\begin{aligned} w_n(\lambda) &= \frac{1}{Z(\beta)}\exp(-\beta E_n) \\ &= \frac{\exp\left(-\beta(an+b)\right)}{\mathrm{Tr}\exp\left(-\beta(an+b)\right)} = C_1(\lambda)\left[C_2(\lambda)\right]^n, \end{aligned} \tag{43}$$

where the constants $\lambda = \lambda(a;b)$ are obtained by solving the corresponding stationary Schrödinger equation. In this case the constants for the quasi-probability P-distribution function are [7]:

$$C_1(\lambda) = 1 - \exp(-\beta a); \qquad C_2(\lambda) = \exp(-\beta a); \qquad \beta = \frac{1}{k_B T}. \tag{44}$$

The last integral equation is also the Stieltjes moment problem similarly as there for finding the weight function of the integration measure. The solution of this problem is of the following kind:

$$R\left(|z|^2;\lambda\right) \equiv \frac{C_1(\lambda)}{C_2(\lambda)} G_{p_2q_2}^{m_2n_2}\left(\left[C_2(\lambda)\right]^{-1}|z|^2 \,\middle|\, \begin{matrix}\{a_{n_2;p_2}\}\\ \{b_{m_2;q_2}\}\end{matrix}\right), \tag{45}$$

where $C_1(\lambda)$ and $C_2(\lambda)$ are some constants, depending on the structure of the probability density function.

Finally, the quasi-probability P-distribution function becomes:

$$P\left(|z|^2;\lambda\right) \equiv \frac{C_1(\lambda)}{C_2(\lambda)} \cdot \frac{G_{p_2q_2}^{m_2n_2}\left([C_2(\lambda)]^{-1}|z|^2 \mid {}^{\{a_{n_2};p_2\}}_{\{b_{m_2};q_2\}}\right)}{G_{p_2q_2}^{m_2n_2}\left(|z|^2 \mid {}^{\{a_{n_2};p_2\}}_{\{b_{m_2};q_2\}}\right)}. \tag{46}$$

The thermal average of an observable A in a mixed state described by the density operator ρ can be calculated as follows:

$$\begin{aligned}\langle A\rangle = \mathrm{Tr}(\rho A) &= \int d\mu\,(\sigma;\lambda)\,\langle\sigma;\lambda|\rho A|\sigma;\lambda\rangle \\ &= \frac{C_1(\lambda)}{C_2(\lambda)} \int \frac{d^2 z}{\pi} \cdot S^{(0)}\left(|z|^2;\lambda\right) \\ &\quad \times G_{p_2q_2}^{m_2n_2}\left([C_2(\lambda)]^{-1}\,|z|^2 \,\middle|\, \begin{matrix}\{a_{n_1;p_1}\}\\ \{b_{m_1;q_1}\}\end{matrix}\right) \\ &\quad \times \langle z;\lambda|A|z;\lambda\rangle,\end{aligned} \tag{47}$$

which, after some calculations, leads to the expression:

$$\begin{aligned}\langle A\rangle = \mathrm{Tr}(\rho A) = \frac{C_1(\lambda)}{C_2(\lambda)} \sum_{n=0}^{\infty} \frac{\langle n;\lambda|A|n;\lambda\rangle}{\rho_n(\lambda)} \int_0^{\infty} d\left(|z|^2\right)\left(|z|^2\right)^n \\ \times G_{p_2q_2}^{m_2n_2}\left([C_2(\lambda)]^{-1}\,|z|^2 \,\middle|\, \begin{matrix} a_1,\ldots,a_{n_2}; & a_{n_2+1},\ldots,a_{p_2} \\ b_1,\ldots,b_{m_2}; & b_{m_2+1},\ldots,b_{q_2}\end{matrix}\right).\end{aligned} \tag{48}$$

This integral can be solved by using the general formula of the Meijer's functions [5]:

$$\begin{aligned}&\int_0^{\infty} dx\, x^{s-1} G_{pq}^{mn}\left(\alpha x \,\middle|\, \begin{matrix} a_1,\ldots,a_n; & a_{n+1},\ldots,a_p \\ b_1,\ldots,b_m; & b_{m+1},\ldots,b_q\end{matrix}\right) \\ &\quad = \frac{1}{\alpha^s} \cdot \frac{\prod_{j=1}^{m}\Gamma(b_j+s)\prod_{j=1}^{n}\Gamma(1-a_j-s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-s)\prod_{j=n+1}^{p}\Gamma(a_j+s)}.\end{aligned} \tag{49}$$

So, we obtain in our case:

$$\begin{aligned}\langle A\rangle = \frac{C_1(\lambda)}{C_2(\lambda)} \sum_{n=0}^{\infty} \frac{\langle n;\lambda|A|n;\lambda\rangle}{\rho_n(\lambda)} [C_2(\lambda)]^{n+1} \\ \times \frac{\prod_{j=1}^{m_2}\Gamma(b_j+1+n)\prod_{j=1}^{n_2}\Gamma(-a_j-n)}{\prod_{j=m+1}^{q_2}\Gamma(-b_j-n)\prod_{j=n+1}^{p_2}\Gamma(a_j+1+n)}.\end{aligned} \tag{50}$$

We can simplify this relatively complicate expression if we observe that in the equation for the function $R(|z|^2;\lambda)$ the integral is equal just with:

$$\begin{aligned}&\int_0^{\infty} d\left(|z|^2\right)\left(|z|^2\right)^n G_{p_2q_2}^{m_2n_2}\left(C_2(\lambda)|z|^2 \,\middle|\, \begin{matrix} a_1,\ldots,a_{n_2}; & a_{n_2+1},\ldots,a_{p_2} \\ b_1,\ldots,b_{m_2}; & b_{m_2+1},\ldots,b_{q_2}\end{matrix}\right) \\ &\quad = [C_2(\lambda)]^{n+1}\, w_n(\lambda)\rho_n(\lambda).\end{aligned} \tag{51}$$

So, finally, we obtain:

$$\langle A\rangle = \sum_{n=0}^{\infty} w_n(\lambda)\langle n;\lambda \mid A \mid n;\lambda\rangle, \tag{52}$$

i.e. the usual expression for the expectation value of observable, a result which was expected because the trace is independent of the basis.

Particularly, for $A = N^s$ we obtain:

$$\begin{aligned}\langle N^s\rangle &= \sum_{n=0}^{\infty} w_n(\lambda)\langle n;\lambda|N^s|n;\lambda\rangle \\ &= \sum_{n=0}^{\infty} w_n(\lambda) n^s.\end{aligned} \tag{53}$$

By analogy, a function of N, say an exponential, has the expectation value:

$$\langle e^{\varepsilon N}\rangle = \sum_{n=0}^{\infty} w_n(\lambda) e^{\varepsilon N}, \tag{54}$$

so, we have:

$$\langle N^s\rangle = \lim_{\varepsilon\to 0}\left(\frac{d}{d\varepsilon}\right)^s \left[\langle e^{\varepsilon N}\rangle\right]. \tag{55}$$

In this manner we can calculate the mixed counterpart of the Mandel's parameter, defined as [3]:

$$\begin{aligned}Q &= \frac{\langle \Delta N^2\rangle - \langle N\rangle}{\langle N\rangle} \\ &= \frac{\langle N^2\rangle - (\langle N\rangle)^2}{\langle N\rangle} - 1,\end{aligned} \tag{56}$$

with the same interpretation relatively to the Poisson distribution as earlier.

4 Application to Some Simple Quantum Systems

In order to illustrate the above assertions we apply the calculations below to some simple quantum systems: the one dimensional quantum oscillator (HO-1D) and the pseudoharmonic oscillator (PHO) for the mixed (thermal) states characterized by two distribution functions: the canonical distribution $w_n^{(th)}(\overline{n}_T)$ and the Poisson distribution $w_n^{(P)}(\langle N\rangle)$.

4.1 The One Dimensional Harmonic Oscillator (HO-1D)

If we take, as a significant example, the one dimensional harmonic oscillator (HO-1D), then the BG-CSs are defined as:

$$a|z\rangle = z|z\rangle \tag{57}$$

and, consequently, their expansion in the Fock vector basis is

$$|z\rangle = \exp\left(-\frac{1}{2}|z|^2\right)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle, \tag{58}$$

where $|z;\lambda=0\rangle \equiv |z\rangle$, $|n;\lambda=0\rangle \equiv |n\rangle$ and the sum is $S^{(0)}(|z|^2;\lambda=0) = \exp(|z|^2)$.

The structure constants of the CSs are $\rho_n(\lambda=0) = n! = \Gamma(n+1)$, so the integral equation for the integration measure weight $g(|z|^2)$ function is

$$\int_0^{\infty} d\left(|z|^2\right)\left(|z|^2\right)^{s-1} g\left(|z|^2\right) = \Gamma(s), \tag{59}$$

whose solution is as follow [8]:

$$g\left(|z|^2\right) = G_{01}^{10}\left(|z|^2 \ |0\right) = \exp\left(-\mid z\mid^2\right), \tag{60}$$

while the integration measure becomes

$$d\mu(z) = \frac{d^2z}{\pi}. \tag{61}$$

Let us recall that for the HO-1D, all CSs are the same, i.e. the BG-CSs are identical with the Klauder-Perelomov coherent states (KP-CSs).

The last are defined by applying the displacement operator on the ground (fiducial) state.

In this case the density of probability of transition between a Fock state $\mid n\rangle$ and a BG-CS of the HO-1D $\mid z\rangle$ is just the Poissonian distribution function:

$$P_{n;|z|^2}^{(\text{HO-1D})} = \exp\left(-|z|^2\right)\frac{(|z|^2)^n}{n!} = P_{n;|z|^2}^{Poissonian}. \tag{62}$$

If we calculate the mean value of the integer powers of number-particle operator, we obtain successively:

$$\begin{aligned}\langle N^s\rangle_{|z\rangle} &= \sum_{n=0}^{\infty} n^s P_{n;|z|^2}^{(\text{HO-1D})}\\ &= \exp\left(-|z|^2\right)\sum_{n=0}^{\infty} n^s\frac{(|z|^2)^n}{n!}\\ &= \exp\left(-|z|^2\right)\left(|z|^2\frac{d}{d|z|^2}\right)^s \exp\left(|z|^2\right).\end{aligned} \tag{63}$$

Particularly, we obtain:

$$\langle N\rangle_{|z\rangle} = |z|^2; \qquad \langle N^2\rangle_{|z\rangle} = |z|^2 + |z|^4;$$
$$\left(\sigma_N^2\right)_{|z\rangle} = |z|^2$$

and the Poisson distribution function for the HO-1D can also be written as:

$$\begin{aligned}P_{n;|z|^2}^{(\text{HO-1D})} &= \exp\left(-\langle N\rangle_{|z\rangle}\right)\frac{(\langle N\rangle_{|z\rangle})^n}{n!}\\ &= \exp\left(-\left(\sigma_N^2\right)_{|z\rangle}\right)\frac{[(\sigma_N^2)_{|z\rangle}]^n}{n!}.\end{aligned} \tag{65}$$

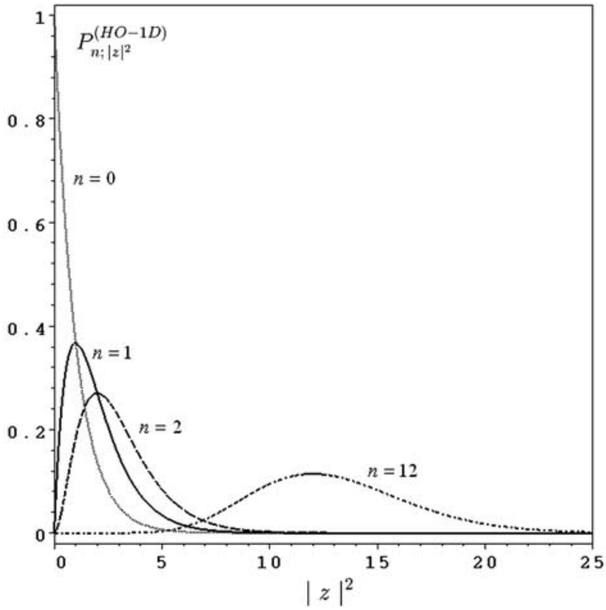


Fig. 1 Dependence of Poissonian distribution on $|z|^2$ with n as a parameter

Consequently, we obtain that the Mandel parameter is $Q_{|z;\lambda\rangle} = 0$, i.e. the field of BG-CSs of the HO-1D is Poissonian. In connection with them, all probability distribution functions are compared with the Poissonian distribution, among them being the sub-Poissonian and supra-Poissonian distributions.

If we represent the Poissonian distribution function $P^{(\text{HO-1D})}_{n;|z|^2}$ as a function of $|z|^2$, as in Fig. 1, the number of particles n playing the role of a parameter, then a maximum is obtained for $|z|^2 = n$.

On the other hand, if we take $|z|^2$ as a parameter and n as a discrete variable, as in Fig. 2, then the maximum value is obtained from:

$$\begin{aligned}\frac{d}{dn}P^{(\text{HO-1D})}_{n;|z|^2} &= \exp\left(-|z|^2\right)\frac{d}{dn}\left[\frac{(|z|^2)^n}{n!}\right] \\ &= \exp\left(-|z|^2\right)\frac{(|z|^2)^n}{(n!)^2}\left[n!\ln|z|^2 - \frac{d}{dn}(n!)\right] \\ &= P^{(\text{HO-1D})}_{n;|z|^2}\cdot\left[\ln|z|^2 - \frac{1}{n!}\frac{d}{dn}(n!)\right] \\ &= P^{(\text{HO-1D})}_{n;|z|^2}\cdot\left[\ln|z|^2 - \frac{d}{dn}(\ln n!)\right].\end{aligned} \tag{66}$$

In order to differentiate the factorial, we have to use a form of Stirling's approximation:

$$\ln n! = n\ln n - n + 1. \tag{67}$$

So, that:

$$\frac{d}{dn}(\ln n!) = \ln n \tag{68}$$

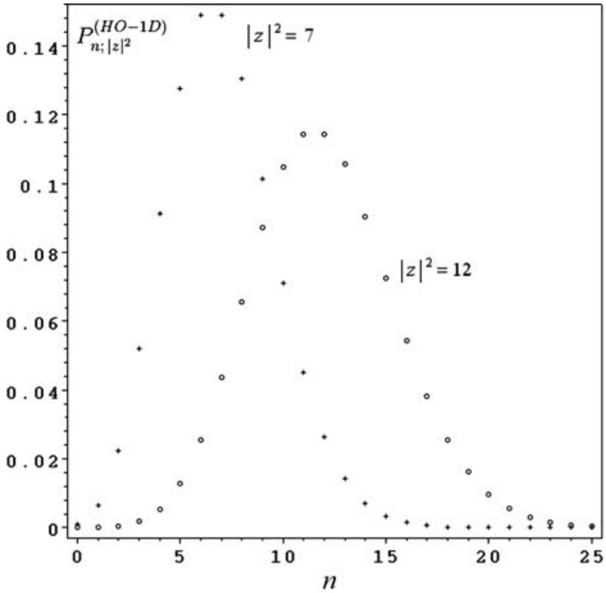


Fig. 2 Dependence of Poissonian distribution on n with $|z|^2$ as a parameter

and finally, we obtain:

$$\frac{d}{dn} P^{\text{(HO-1D)}}_{n;|z|^2} = P^{\text{(HO-1D)}}_{n;|z|^2} \cdot \ln \frac{|z|^2}{n}. \tag{69}$$

The maximum value of the Poisson distribution function $P^{\text{(HO-1D)}}_{n;|z|^2}$ is obtained by equalizing with zero the above derivative which is finally reduced to the condition:

$$n = |z|^2. \tag{70}$$

On the other hand, by using the following form of the Stirling's approximation (which is valid for great numbers n), i.e.:

$$n! \cong n^n e^{-n} \sqrt{2\pi n}, \tag{71}$$

the maximal value of the Poisson distribution function, as function on n, is:

$$P^{\text{(HO-1D)}}_{n;|z|^2} |_{\max} = \exp\left(-|z|^2\right) \frac{(|z|^2)^n}{n!} \cong \frac{1}{\sqrt{2\pi n}}, \tag{72}$$

which is also shown in Fig. 2.

The Thermal Distribution Function If we consider the mixed states, particularly the thermal states of the HO-1D, we have, for $\lambda = \beta$, the following distribution function:

$$w^{\text{(HO-1D)}}_n(\beta) = \frac{1}{Z(\beta)} \exp\left[-\beta\hbar\omega\left(n + \frac{1}{2}\right)\right] = \frac{1}{\overline{n}_T + 1}\left(\frac{\overline{n}_T}{\overline{n}_T + 1}\right)^n, \tag{73}$$

where

$$\overline{n}_T = \frac{1}{e^{\beta\hbar\omega} - 1} \equiv \frac{1}{e^{\frac{T_E}{T}} - 1} \tag{74}$$

is the Bose distribution function, while $T_E = \frac{\hbar\omega}{k_B}$ is the well-known Einstein-Debye characteristic temperature.

This distribution function ensures that the corresponding density operator will be normalized:

$$\rho_{th}^{\text{HO-1D}}(\beta) = \frac{1}{Z(\overline{n}_T)} \sum_{n=0}^{\infty} \left(\frac{\overline{n}_T}{\overline{n}_T + 1} \right)^{n+\frac{1}{2}} |n\rangle\langle n|$$

$$= \frac{1}{\overline{n}_T + 1} \sum_{n=0}^{\infty} \left(\frac{\overline{n}_T}{\overline{n}_T + 1} \right)^{n} |n\rangle\langle n|, \tag{75}$$

$$\text{Tr}\rho = 1; \qquad Z(\overline{n}_T) = \sqrt{\overline{n}_T(\overline{n}_T + 1)}. \tag{76}$$

Finally, for thermal states of the HO-1D we have the following expectation values for the integer powers of the particle-number operator:

$$\langle N^s \rangle_{th}^{\text{HO-1D}} = \frac{1}{\overline{n}_T + 1} \sum_{n=0}^{\infty} \left(\frac{\overline{n}_T}{\overline{n}_T + 1} \right)^{n} n^s$$

$$= \frac{1}{\overline{n}_T + 1} \left[\overline{n}_T (\overline{n}_T + 1) \frac{d}{d\overline{n}_T} \right]^s (\overline{n}_T + 1). \tag{77}$$

Consequently, the Mandel's parameter is

$$Q_{th}^{\text{HO-1D}} = \overline{n}_T. \tag{78}$$

We can observe that at any temperature T the Mandel's parameter $Q_{th}^{\text{HO-1D}} > 0$, so the corresponding thermal field is supra-Poissonian.

By analogy, the integral equation for function $R(|z|^2)$ is

$$\int_0^{\infty} d\left(|z|^2\right) \left(|z|^2\right)^{s-1} R\left(|z|^2\right) = \frac{1}{\overline{n}_T} \frac{1}{(\frac{\overline{n}_T+1}{\overline{n}_T})^s} \Gamma(s), \tag{79}$$

whose solution is [8]

$$R\left(|z|^2\right) = \frac{1}{\overline{n}_T} G_{01}^{10} \left(\frac{\overline{n}_T + 1}{\overline{n}_T} |z|^2 \mid 0 \right)$$

$$= \frac{1}{\overline{n}_T} \exp\left(-\frac{\overline{n}_T + 1}{\overline{n}_T} |z|^2 \right). \tag{80}$$

Finally, the P-quasi-distribution function for the thermal state of the HO-1D becomes:

$$P_{th}^{(\text{HO-1D})}\left(|z|^2; \overline{n}_T\right) = \frac{1}{\overline{n}_T} \exp\left(-\frac{1}{\overline{n}_T} |z|^2 \right). \tag{81}$$

We can observe from the Fig. 3 that this function is a (semi) Gauss distribution function with the variable $|z|^2$ (which is defined only on the positive real half-plane).

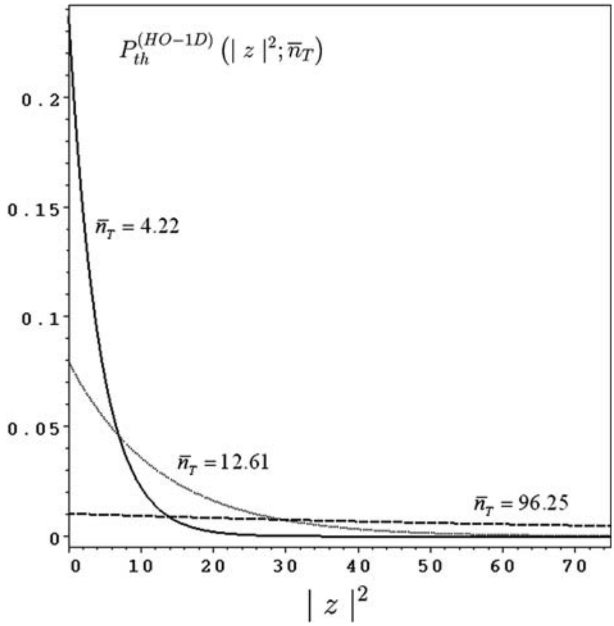


Fig. 3 Dependence of thermal distribution function on $|z|^2$ with $\overline{n}_T$ as a parameter

The Poisson Distribution Function Let us consider a chaotic light field which is characterized by the Poisson distribution function

$$w_n^{(P)}(\langle N\rangle) = \exp(-\langle N\rangle)\frac{(\langle N\rangle)^n}{n!}. \tag{82}$$

In the Fock basis vectors of the HO-1D, the density operator is

$$\rho_P^{(\text{HO-1D})}(\langle N\rangle) = \sum_{n=0}^{\infty}\exp(-\langle N\rangle)\frac{(\langle N\rangle)^n}{n!}|n\rangle\langle n|. \tag{83}$$

Evidently, the differences appear only in the case of mixed states. So, the equation for the function $R(x)$ is

$$\int_0^\infty d\left(|z|^2\right)\left(|z|^2\right)R\left(|z|^2\right) = \frac{\exp(-\langle N\rangle)}{\langle N\rangle}\frac{1}{(\frac{1}{\langle N\rangle})^s}. \tag{84}$$

This is a particular case of the Stieltjes moment problem and its solution is [9]

$$\begin{aligned} R(x) &= \frac{\exp(-\langle N\rangle)}{\langle N\rangle}G_{00}^{00}\left(\frac{1}{\langle N\rangle}x\middle|\right) \\ &= \frac{\exp(-\langle N\rangle)}{\langle N\rangle}\delta\left(\frac{1}{\langle N\rangle}x-1\right). \end{aligned} \tag{85}$$

Using the properties of the delta Dirac function, finally, we obtain:

$$R(x) = \exp(-\langle N\rangle)\,\delta(x-\langle N\rangle), \tag{86}$$

so that the P-quasi-distribution function is

$$P_P^{(\text{HO-1D})}\left(|z|^2\right) = \exp\left(|z|^2-\langle N\rangle\right)\delta\left(|z|^2-\langle N\rangle\right). \tag{87}$$

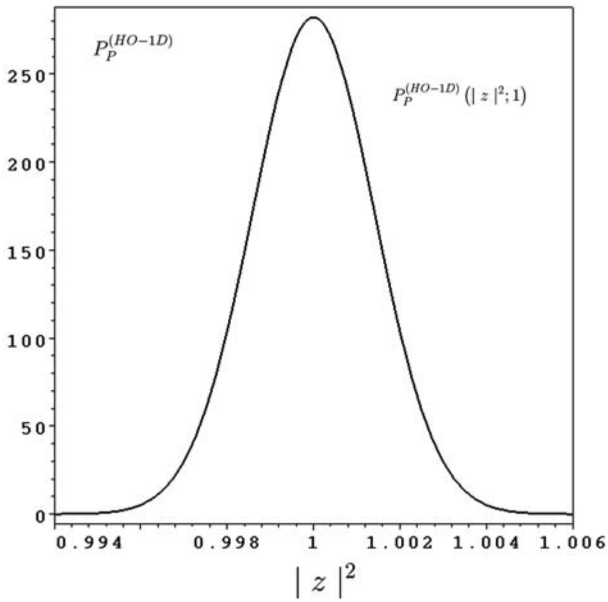


Fig. 4 Dependence of Poisson distribution function $P_P^{(\text{HO-1D})}$ on $|z|^2$ with $\langle N\rangle = 1$ as a parameter

In Fig. 4 we have represented the dependence of the Poisson distribution $P_P^{(\text{HO-1D})}$ as a function of $|z|^2$ with $\langle N\rangle$ as a parameter. It is not difficult to verify that this function is normalized to unity.

4.2 The Pseudoharmonic Oscillator (PHO)

The Barut-Girardello coherent states of this oscillator are defined as the eigenstates of the lowering operator K_- which is one of the generators of $SU(1,1)$ quantum group associated with this oscillator [10]:

$$K_-|z;k\rangle = z|z;k\rangle, \tag{88}$$

where the real number k is called the Bargmann index and labels the irreducible representations of this group. We are only interested in the unitary irreducible representations known as positive discrete series, where $k > 0$. For the PHO these states were firstly derived in [10] and their decomposition in the Fock vector basis reads:

$$|z;k\rangle = \sqrt{\frac{|z|^{2k-1}}{I_{2k-1}(|z|)}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\Gamma(n+1)\Gamma(n+2k)}}|n;k\rangle, \tag{89}$$

where $I_{2k-1}(x)$ are the Bessel functions and $\Gamma(x)$-the Euler's gamma functions. In the PHO model the rotational and vibrational motions are decoupled, and as a consequence the energy eigenvalues are linear in the vibration quantum number n (while the dependence on the rotational quantum number J is more complicated) [11]:

$$E_{nJ} = \hbar\omega k - \frac{1}{4}m_{red}\omega^2 r_0^2 + \hbar\omega n \equiv E_{0J} + \hbar\omega n, \tag{90}$$

where m_{red}, ω and r_0 are, respectively, the reduced mass, angular frequency and the equilibrium position of PHO. The dependence of the Bargmann index on the rotational quantum number is :

$$k = \frac{1}{2} + \frac{1}{2}\sqrt{\left(J+\frac{1}{2}\right)^2 + \frac{m_{red}\omega}{2\hbar} r_0^4}. \tag{91}$$

The unitary operator decomposition

$$\int d\mu(z;k)|z;k\rangle\langle z;k| = 1 \tag{92}$$

is realised if the integration measure is of the following kind [10]:

$$d\mu(z;k) = \frac{d^2 z}{\pi} 2K_{2k-1}\left(2|z|\right) I_{2k-1}\left(2|z|\right). \tag{93}$$

This expression can be easily obtained if we apply the considerations from the Sect. 2 about the finding of the function $g(x)$.

The Thermal Distribution Function Due to the structure of the energy spectrum of the PHO, the thermal distribution function $w^{(th)}(\overline{n}_T)$ is identical with those for the HO-1D:

$$\begin{aligned} w_n^{(th)}(\overline{n}_T) &= \frac{1}{\mathrm{Tr}[\exp(-\beta E_{nJ})]} \exp(-\beta E_{nJ}) \\ &= \frac{1}{\overline{n}_T + 1}\left(\frac{\overline{n}_T}{\overline{n}_T + 1}\right)^n, \end{aligned} \tag{94}$$

so, the corresponding normalized density operator is

$$\rho_{th}^{PHO}(\beta) = \frac{1}{\overline{n}_T + 1}\sum_{n=0}^{\infty}\left(\frac{\overline{n}_T}{\overline{n}_T + 1}\right)^n |n;k\rangle\langle n;k|. \tag{95}$$

In order to find the corresponding $R(x)$ function, we have to solve the following Stieltjes moment problem:

$$\int_0^\infty d\left(|z|^2\right)\left(|z|\right)^{s-1} R\left(|z|^2\right) = \frac{1}{\overline{n}_T}\frac{1}{\left(\frac{\overline{n}_T+1}{\overline{n}_T}\right)^s} \times \Gamma(s)\Gamma(2k-1+s), \tag{96}$$

whose solution is [5, 9]

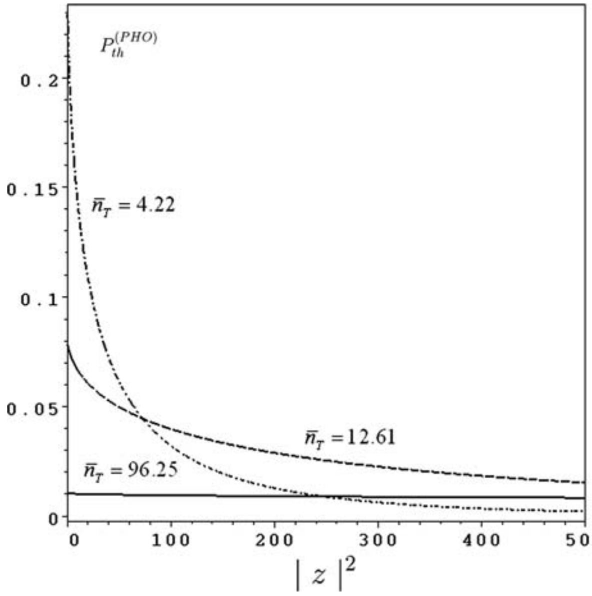
$$\begin{aligned} R\left(|z|^2\right) &= \frac{1}{\overline{n}_T} G_{02}^{20}\left(\frac{\overline{n}_T+1}{\overline{n}_T}|z|^2 \,\middle|\, \begin{matrix} 2k-1 \\ 0 \end{matrix}\right) \\ &= 2\frac{1}{\overline{n}_T}\left(\frac{\overline{n}_T+1}{\overline{n}_T}|z|^2\right)^{k-\frac{1}{2}} K_{2k-1}\left(2\sqrt{\frac{\overline{n}_T+1}{\overline{n}_T}}|z|\right). \end{aligned} \tag{97}$$

This lead to the following P-quasi-distribution function [3, 8]:

$$P_{th}^{(PHO)}\left(|z|^2\right) = \frac{1}{\overline{n}_T}\left(\frac{\overline{n}_T+1}{\overline{n}_T}\right)^{k-\frac{1}{2}} \frac{K_{2k-1}(2\sqrt{\frac{\overline{n}_T+1}{\overline{n}_T}}|z|)}{K_{2k-1}(2|z|)}. \tag{98}$$

In Fig. 5 we have represented the dependence of thermal distribution $P_{th}^{(PHO)}$ considered as a function on $|z|^2$ with $\overline{n}_T$ as a parameter.

Fig. 5 Dependence of thermal distribution $P_{th}^{(PHO)}$ on $|z|^2$ with $\overline{n}_T$ as a parameter



The Poisson Distribution Function The density operator of the mixed states of the PHO with the Poisson distribution function is

$$\rho_P^{(PHO)}(\langle N\rangle) = \exp(-\langle N\rangle)\sum_{n=0}^{\infty}\frac{(\langle N\rangle)^n}{n!}|n;k\rangle\langle n;k|, \tag{99}$$

while the diagonal representation of this operator in the BG-CSs basis of PHO and Poisson distribution function reads:

$$\rho_P^{(PHO)}(\langle N\rangle) = \int d\mu(z;k) \mid z;k\rangle P_P^{(PHO)}\left(|z|^2;k\right)\langle z;k|. \tag{100}$$

The moment problem equation for the corresponding $R(x)$ function for PHO is

$$\int_0^\infty d\left(|z|^2\right)\left(|z|^2\right)^{s-1} R\left(|z|^2\right) = \frac{\exp(-\langle N\rangle)}{\langle N\rangle}\frac{1}{(\frac{1}{\langle N\rangle})^s}\Gamma(2k-1+s), \tag{101}$$

whose solution becomes [5, 9]

$$\begin{aligned} R\left(|z|^2\right) &= \frac{\exp(-\langle N\rangle)}{\langle N\rangle} G_{01}^{10}\left(\frac{1}{\langle N\rangle}|z|^2 \;\middle|\; 2k-1\right) \\ &= \frac{\exp(-\langle N\rangle)}{\langle N\rangle}\left(\frac{1}{\langle N\rangle}|z|^2\right)^{2k-1}\exp\left(-\frac{1}{\langle N\rangle}|z|^2\right). \end{aligned} \tag{102}$$

Finally, the P-quasi-distribution function becomes

$$P_P^{(PHO)}\left(|z|^2;k\right) = \frac{1}{2}\frac{\exp(-\langle N\rangle)}{(\langle N\rangle)^{2k}}\frac{\exp(-\frac{1}{\langle N\rangle}|z|^2)}{K_{2k-1}(2|z|)}. \tag{103}$$

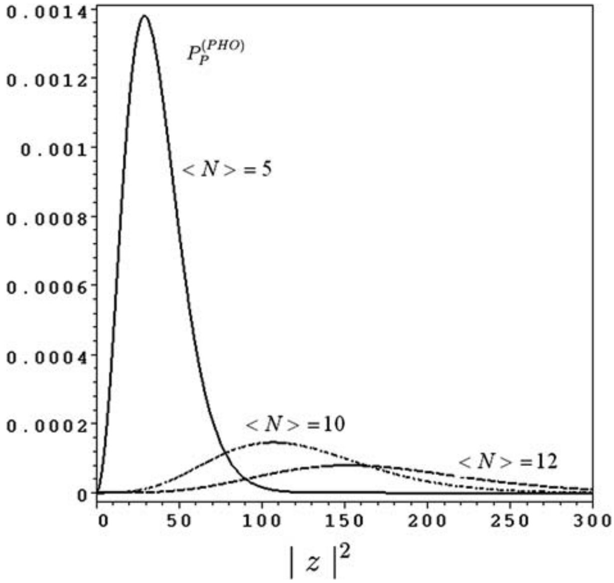


Fig. 6 Dependence of Poisson distribution $P_P^{(PHO)}$ for PHO, as a function on $|z|^2$, with $\langle N\rangle$ as the parameter

The form of Poisson distribution $P_P^{(PHO)}$ for PHO, as a function on $|z|^2$, with parameter $\langle N\rangle$ is represented in Fig. 6.

This is apparently a complicate expression, so we have to verify if it accomplishes the normalization relation:

$$\int d\mu(z;k)P^{(PHO)}\left(|z|^2\right)=1. \tag{104}$$

After a straightforward calculations, for which we have to use the following equations [12]:

$$\int_0^\infty dy y^{\nu+1}e^{-\alpha y^2}J_\nu(\beta y)=\frac{\beta^\nu}{(2\alpha)^{\nu+1}}\exp\left(-\frac{\beta^2}{4\alpha}\right), \tag{105}$$

$$J_\nu(iz)=i^\nu I_\nu(z), \tag{106}$$

we obtain that this relation is fulfilled, which shows us that the obtained expression for the quasi-distribution function P is correct.

Now let us to calculate the expectation values of the mixed states PHO with the Poisson distribution function. We begin with the integer powers of the number particle operator:

$$\begin{aligned}\langle N^s\rangle &= \mathrm{Tr}\left(N^s\rho_P^{(PHO)}(\langle N\rangle)\right)\\ &=\exp(-\langle N\rangle)\sum_{n=0}^{\infty}\frac{(\langle N\rangle)^n}{n!}N^s\\ &=\exp(-\langle N\rangle)\left(\langle N\rangle\frac{d}{d\langle N\rangle}\right)^s\exp(\langle N\rangle).\end{aligned} \tag{107}$$

By using this equation it is easy to calculate that the Mandel's parameter for the PHO with the Poisson distribution function is

$$Q_P^{(PHO)} = 0, \tag{108}$$

i.e. all mixed states of the PHO with the Poisson distribution function are the Poissonian states, as it is to be expected.

5 Conclusions and Outlook

The generalized Barut-Girardello coherent states we have constructed satisfy all Klauder's minimal prescriptions imposed for coherent states [4]. In order to construct these coherent states it is necessary only to know the action of the general lowering operator A_-, i.e. to know the functions $e_j(\lambda)$. So, implicitly, we will be able to construct the structure constants $\rho_n(\lambda)$ of the generalized BG-CSs.

The role of the structure constants $\rho_n(\lambda)$ is fundamental: they determine the specific expression of some CS and make difference between different CSs. These differences are transferred also to the case of mixed states, i.e. in the expressions of the Husimi's Q-function, quasi-probability P-distribution function and also the mixed counterpart of Mandel's parameter.

Evidently, that different probability distribution functions $w_n(\lambda)$ lead to different expressions of the Q and P-functions.

So, the sequence of the dependence can be presented as follows:

– Pure CSs: $e_j(\lambda) \rightarrow \rho_n(\lambda) \rightarrow |z;\lambda\rangle \rightarrow \rho_{|z;\lambda\rangle}$.
– Mixed CSs: $w_n(\lambda) \longrightarrow \rho(\lambda) \rightarrow |z;\lambda\rangle \rightarrow P(|z|^2;\lambda)$, Q.

In conclusion, our presented approach, based in part on the use of the Meijer's G-functions allows, on the one part a general construction of generalized Barut-Girardello coherent states and one the other hand shows the differences between the CSs and their properties in the case of various Hamiltonians and arbitrary distribution functions for mixed states.

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